

New table of Bounds on Permutation Codes under Kendall τ -Metric

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Abstract—In order to overcome the challenges posed by flash memories, the rank modulation scheme was proposed. In the rank modulation the codewords are permutations. In this paper, we study permutation codes with a specified length and minimum Kendall τ -distance, and with as many codewords (permutations) as possible. We managed to make many significant improvements in the size of the best known codes. In particular, we show that for all $n \geq 6$ and for all $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$ the largest size of a permutation code of length n and minimum distance at least d under Kendall τ -metric is 4.

Index Terms—Rank modulation, Kendall τ -Metric, permutation codes.

I. INTRODUCTION AND RESULTS

In order to overcome the challenges posed by flash memories, the rank modulation scheme was proposed [5]. In the rank modulation framework the codewords are permutations. In this context, permutation codes were chiefly studied under three metrics, the Kendall τ -metric [1], [5], [8], [10], [11], the Ulam metric [7] and the l_∞ metric [6], [9]. In this paper, we focus on the permutation codes under the Kendall τ -metric.

Let n be a positive integer and S_n denotes the symmetric group on n letters, i.e., the set of all $n!$ permutations of $[n] := \{1, \dots, n\}$. By a permutation code (PC) of length n we simply mean a non-empty subset of S_n . Throughout for a permutation $\pi \in S_n$, we use the vector notation of π that is $[\pi(1), \pi(2), \dots, \pi(i), \pi(i+1), \dots, \pi(n)]$. For two permutations π and σ , their composition, denoted by $\sigma\pi$, is the permutation with $\sigma\pi(i) = \pi(\sigma(i))$ for all $i \in [n]$. The identity element of S_n is denoted by $\xi := [1, 2, \dots, n]$.

Given a permutation $\pi \in S_n$, an adjacent transposition, $(i, i+1)$, for some $1 \leq i \leq n-1$, applied to π on left will results in the permutation $[\pi(1), \pi(2), \dots, \pi(i+1), \pi(i), \dots, \pi(n)]$. For two permutations $\rho, \sigma \in S_n$, the Kendall τ -distance between ρ and σ , $d_K(\rho, \sigma)$, is defined as the minimum number of adjacent transpositions needed to write $\rho\sigma^{-1}$ as their product.

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There is a well-known equivalent expression for $d_K(\rho, \sigma)$ [5] as follows:

$$d_K(\rho, \sigma) = |\{(i, j) \mid \rho^{-1}(i) < \rho^{-1}(j) \wedge \sigma^{-1}(j) > \sigma^{-1}(i)\}|.$$

A permutation code $\mathcal{C}$ of length n is called an (n, d) -PC, if $d_K(\pi, \sigma) \geq d$ for each distinct elements $\pi, \sigma \in \mathcal{C}$. The largest size of an (n, d) -PC is denoted by $P(n, d)$ and a PC attaining this size is said to be optimal. A central question in the theory of permutation codes is the determination of the value of $P(n, d)$, or finding good lower and upper bounds on $P(n, d)$. Several researchers have presented bounds on $P(n, d)$ (see [2], [3], [5], [8], [10], [11]). It is known that $P(n, 1) = n!$, $P(n, 2) = \frac{n!}{2}$ and if $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$, then $P(n, d) = 2$ (see [3, Theorem 10]). Also, the exact value of $P(n, d)$ is determined for $n = 5$ and $n = 6$ when $d \neq 3$ [8]. In this paper, we determine the exact value of $P(n, d)$ for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$ as follows:

Theorem I.1. $P(n, d) = 4$, for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$.

To prove Theorem I.1, we need the following simple lemma:

Lemma I.2. Let $n \geq 5$ be an integer. If $n \equiv 0, 2 \pmod{3}$ ($n \equiv 1 \pmod{3}$), then there exist 3 non-empty subsets with the same sumset which partitions $[n]$ ($[n] \setminus \{1\}$), respectively.

Proof. If n is 5, 6, 7, 8, 9 and 10, respectively, then $\{\{5\}, \{1, 4\}, \{3, 2\}\}, \{\{6, 1\}, \{5, 2\}, \{3, 4\}\}, \{\{2, 7\}, \{3, 6\}, \{4, 5\}\}, \{\{8, 4\}, \{7, 3, 2\}, \{1, 5, 6\}\}, \{\{6, 5, 4\}, \{9, 1, 2, 3\}, \{8, 7\}\}$ and $\{\{10, 8\}, \{9, 2, 7\}, \{3, 4, 6, 5\}\}$ are the partitions of $[n]$ or $[n] \setminus \{1\}$ satisfying the lemma. Now suppose that $n > 10$. Hence there exist $t > 0$ and $r \in \{5, 6, 7, 8, 9, 10\}$ such that $n = 6t + r$. Note that if $n \equiv 1 \pmod{3}$, then $r \in \{7, 10\}$. Consider $t + 1$ subsets $\Theta_1, \dots, \Theta_{t+1}$ of $[n]$ as follows:

$$\underbrace{1, \dots, r}_{\Theta_1}, \underbrace{r+1, \dots, r+6}_{\Theta_2}, \dots, \underbrace{n-11, \dots, n-6}_{\Theta_t}, \underbrace{n-5, \dots, n}_{\Theta_{t+1}}$$

Clearly, for all $2 \leq i \leq t+1$, there exists $k \in [n]$ such that $\Theta_i = \{k+1, k+2, \dots, k+6\}$. Hence, 3 sets $\Theta_{i1} := \{k+1, k+6\}$, $\Theta_{i2} := \{k+2, k+5\}$ and $\Theta_{i3} := \{k+3, k+4\}$ with the same sumset partition the set Θ_i . Therefore, since $r \in \{5, 6, 7, 8, 9, 10\}$, corresponding to each set Θ_i , $1 \leq i \leq t+1$, there exists a partition of 3 sets Θ_{i1} , Θ_{i2} and Θ_{i3} with the same sumset for Θ_i . Let $\Delta_j := \bigcup_{i=1}^{t+1} \Theta_{ij}$ for all $j \in \{1, 2, 3\}$. So Δ_1 , Δ_2 and Δ_3 with the same sumset partition $[n]$ or $[n] \setminus \{1\}$ if $n \equiv 0, 2 \pmod{3}$ or $n \equiv 1 \pmod{3}$, respectively. This completes the proof. $\square$

Proof of Theorem I.1. It follows from [10, Theorem 23] that if $P(n, d) \geq 5$, then we must have $\binom{5}{2}d \leq 6 \times \binom{n}{2}$ and therefore $d \leq \frac{3}{5}\binom{n}{2}$. So for all $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$, $P(n, d) \leq 4$. Since $P(n, d+1) \leq P(n, d)$, it is enough to show that there exists an $(n, \lfloor \frac{2}{3}\binom{n}{2} \rfloor)$ -PC of size 4.

Let $N := \sum_{i=1}^{n-1} i = \binom{n}{2}$. It follows from Lemma I.2 that there exist pairwise disjoint subsets Δ_1, Δ_2 and Δ_3 of $[n-1]$ or $[n-1] \setminus \{1\}$ such that if $n-1 \equiv 0, 2 \pmod{3}$ or $n-1 \equiv 1 \pmod{3}$, respectively, then $\sum_{j \in \Delta_i} j = \frac{N}{3}$ or $\sum_{j \in \Delta_i} j = \frac{N-1}{3}$, for all $i \in \{1, 2, 3\}$. Now, suppose that for $n \geq 6$, the subsets Δ_1, Δ_2 and Δ_3 of $[n-1]$ are determined. Corresponding to each Δ_i , we introduce a permutation α_i as follows: let $r_i := |\Delta_i|$, $\Delta'_i := \{n-j \mid j \in \Delta_i\}$ and $\Theta_i := [n] \setminus \Delta'_i$. Suppose that $j_1 < j_2 < \dots < j_{r_i}$ and $l_0 < l_1 < \dots < l_{n-r_i-1}$ are all elements of Δ'_i and Θ_i , respectively. Let $\alpha_i \in S_n$ such that $\alpha_i(t) = j_t$ and $\alpha_i(n-s) = l_s$ for all $t \in \{1, \dots, r_i\}$ and $s \in \{0, \dots, n-r_i-1\}$. Let α_x and α_y be two distinct permutations corresponding to distinct subsets Δ_x and Δ_y , $x, y \in \{1, 2, 3\}$. In view of the definition of α_x , if $i < j$ are two elements of $[n]$, then $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ if and only if $i \in \Delta'_x$. So, since $\Delta'_x \cap \Delta'_y = \emptyset$, we have $(i, j) \in [n]^2$ satisfies $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ and $\alpha_y^{-1}(i) > \alpha_y^{-1}(j)$, if and only if $(i, j) \in A \cup B$, where $A := \{(i, j) \mid i < j, i \in \Delta'_x\}$ and $B := \{(i, j) \mid i > j, j \in \Delta'_y\}$. Hence

$$\begin{aligned} d_K(\alpha_x, \alpha_y) &= |\{(i, j) \mid \alpha_x^{-1}(i) < \alpha_x^{-1}(j) \wedge \alpha_y^{-1}(j) > \alpha_y^{-1}(i)\}| \\ &= |A \cup B| = |A| + |B|. \end{aligned}$$

Therefore, $d_K(\alpha_x, \alpha_y) = \sum_{i \in \Delta_x} i + \sum_{i \in \Delta_y} i$ and so $d_K(\alpha_x, \alpha_y)$ is equal to $\frac{2N}{3}$ if $n-1 \equiv 0, 2 \pmod{3}$ and otherwise is equal to $\frac{2(N-1)}{3} = \lfloor \frac{2}{3}N \rfloor$. Also it is easy to see that

$$\begin{aligned} d_K(\xi, \alpha_x) &= |\{(i, j) \mid i < j \wedge \alpha_x^{-1}(i) > \alpha_x^{-1}(j)\}| \\ &= |\{(i, j) \mid i < j, i \in \Theta_x\}|, \end{aligned}$$

and therefore $d_K(\xi, \alpha_x)$ is equal to $N - \frac{N}{3} = \frac{2}{3}N$ if $n-1 \equiv 0, 2 \pmod{3}$ and is equal to $N - \frac{N-1}{3} = \frac{2N+1}{3} > \lfloor \frac{2}{3}N \rfloor$ if $n-1 \equiv 1 \pmod{3}$. Hence, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is an $(n, \lfloor \frac{2}{3}\binom{n}{2} \rfloor)$ -PC of size 4. This completes the proof.

Example I.3. Let $n = 14$. Suppose $\Delta_1 := \{2, 7, 8, 13\}$, $\Delta_2 := \{3, 6, 9, 12\}$ and $\Delta_3 := \{4, 5, 10, 11\}$. Then $\Delta_1, \Delta_2, \Delta_3$ have the same sumset 30, partitions $[n-1] \setminus \{1\}$. Hence by the proof of Theorem I.1, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is a $(14, 60)$ -PC, where

$$\alpha_1 = [1, 6, 7, 12, 14, 13, 11, 10, 9, 8, 5, 4, 3, 2],$$

$$\alpha_2 = [2, 5, 8, 11, 14, 13, 12, 10, 9, 7, 6, 4, 3, 1],$$

$$\alpha_3 = [3, 4, 9, 10, 14, 13, 12, 11, 8, 7, 6, 5, 2, 1].$$

In Table I the best known bounds on $P(n, d)$, $n \in \{5, 6, 7, 8\}$ are presented. As stated in the key of the table, the bold entries of table shows some results of the current paper. The new lower bounds on $P(n, d)$, $n \in \{7, 8\}$, obtained by the construction of some new codes from the subgroups of S_n and some of their left cosets. In fact, for any pairs of integers (n, d) , we design an algorithm that for each subgroup H of S_n which is an (n, d) -code, can determine a subset S_H of a left transversal set of H in S_n such that $\cup_{x \in S_H \cup \{\xi\}} xH$ is an

(n, d) -code, where $xH := \{xh \mid h \in H\}$. We recall that O_H is called a left transversal set of H in S_n if $S_n = \cup_{x \in O_H} xH$, and $xH \cap x'H = \emptyset$ for all distinct elements $x, x' \in O_H$. Using GAP [4] and by this algorithm we can find new (n, d) -PCs under the Kendall τ -metric improving the lower bounds of $P(n, d)$ for $n \in \{7, 8\}$.

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n/d		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22– 28
5	LB=UB	20^i	12^i	6^i	5^i	2^c	2^c	2^c	2^c	–	–	–	–	–	–	–	–	–	–	–	–
6	UB	116^b	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c	–	–	–	–	–	–	–	–	–
	LB	102^i	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c	–	–	–	–	–	–	–	–	–
7	UB	716^b	420^c	186^a	120^c	66^a	45^c	28^a	21^c	12^g	8^g	4^g	4^g	2^c	2^c	2^c	2^c	2^c	2^c	2^c	–
	LB	–	315	126	84	42	28	15	12	8	7	4	4	–	–	–	–	–	–	–	–
	OLB	588^c	294^f	110^d	55^c	34^d	17^f	14^d	7^a	2^a	2^a	2^a	2^a	2^c	2^c	2^c	2^c	2^c	2^c	2^c	–
8	UB	5039^c	2880^c	1152^a	720^c	363^a	242^c	141^a	99^c	64^a	47^c	32^a	25^c	10^g	8^g	4^g	4^g	2^c	2^c	2^c	2^c
	LB	3696	2184	672	392	168	112	48	48	24	24	14	14	8	8	4	4	–	–	–	–
	OLB	2688^h	1344^a	142^a	76^a	33^a	20^a	12^a	7^a	6^a	4^a	3^a	3^a	1^a	1^a	1^a	1^a	2^c	2^c	2^c	2^c

Key to the superscripts used in Table

superscript a	Sphere packing bound
superscript b	Sphere packing bound+[1, Theorem 3.5]
superscript c	[3, Corollary 5 or Theorems 10,12 or 13]
superscript d	Lower bounds from [5]
superscript f	[5, Theorem 21]
superscript g	[10, Theorem 23]
superscript h	$P(n, 3) \geq \frac{n!}{2^{n-1}}$ [5]
superscript i	[8, Table II]
superscript j	[10, Theorems 18 and 19]
an entry in bold	Results of the current paper

TABLE I: Best known lower bound (LB) and upper bound (UP) on $P(n, d)$.