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IMPROVED UPPER BOUNDS ON THE SIZE OF PERMUTATION CODES UNDER KENDALL τ-METRIC

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Abstract. A central question in the theory of permutation codes is determining the value of $P(n,d)$, representing the size of the largest subset of the set of all permutations on $\{1,\dots,n\}$, S_n, with minimum Kendall τ-distance d. In this paper, we present some of our results regarding the exact values or upper bounds for $P(n,d)$.

1. Introduction and preliminaries

To tackle challenges in flash memories, the rank modulation scheme was introduced, as detailed in [6], employing permutations as codewords. In this context, permutation codes underwent thorough examination utilizing three metrics: the Kendall τ-metric [1, 6, 12, 10, 11], the Ulam metric [8], and the ℓ_∞ metric [7, 9]. This study distinctly focuses on permutation codes under the Kendall τ-metric.

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A Permutation Code (PC) with length n signifies a non-empty subset of S_n, encompassing all permutations of $[n] := \{1, 2, \ldots, n\}$. In the context of a permutation $\pi := [\pi(1), \pi(2), \ldots, \pi(i), \pi(i+1), \ldots, \pi(n)] \in S_n$, an adjacent transposition, denoted as $(i, i+1)$ for $1 \leq i \leq n-1$, transforms π into the permutation $[\pi(1), \pi(2), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n)]$. The Kendall τ-distance between two permutations, ρ and π in S_n, is defined as the minimum number of adjacent transpositions required to express $\rho\pi^{-1}$ as their product. In the context of the Kendall τ-metric, a PC of length n with minimum distance d can correct up to $\frac{d-1}{2}$ errors induced by charge-constrained errors, as cited in [6].

In the realm of permutation code theory, a central question revolves around determining $P(n, d)$, representing the size of the largest code in S_n with minimum Kendall τ-distance d, for $d \leq \binom{n}{2}$. It is known that $P(n, 1) = n!$ and $P(n, 2) = \frac{n!}{2}$. Also it is known that if $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$, then $P(n, d) = 2$ (see [4, Theorem 10]). However, determining $P(n, d)$ turns out to be difficult for $3 \leq d \leq \frac{2}{3}\binom{n}{2}$ and several researchers have presented bounds on $P(n, d)$ (see [3, 4, 6, 12, 10, 11]).

The sphere packing bound [6, Theorems 12 and 13], establishes that $P(n, 3) \leq (n-1)!$. A PC of size $(n-1)!$ and with minimum Kendall τ-distance 3 in S_n is called a 1-perfect code. Notably, in [5, Corollary 2.5 and Theorem 2.6] or [4, Corollary 2], the following result corresponding to the non-existence of 1-perfect codes in S_n is proved:

Theorem 1.1. *If $n > 4$ is a prime number or $4 \leq n \leq 10$, then there is no 1-perfect code in S_n*

The enhancement provided by Theorem 1.1 to the corresponding upper bound of $P(n, 3)$ is modest, improving it by just one. Yet, within this paper, we present some of our results that significantly improve the upper bound of $P(n, 3)$ obtained by Theorem 1.1.

2. MAIN RESULTS

In [1], using a method that is based on the representation theory of symmetric groups, we formulate an integer programming problem depending on the choice of a non-trivial subgroup of S_n. The optimal value of the objective function, obtained through this formulation, serves as an upper bound for $P(n, 3)$ (see [1, Theorem 2.14]). Subsequently, solving the integer programming problem for specific subgroups of S_n results in a reduction of the known upper bound on $P(n, 3)$ by 3, 3, 9, 11, 1, 1, 4 when $n = 6, 7, 11, 13, 14, 15, 17$, respectively. Additionally, this process leads to an enhancement of the upper bound on $P(n, 3)$ for prime values of n as follows:

n	Theorem 1.1	Theorem 2.1	Theorem 2.2
31	1	13	9
37	1	15	62
41	1	16	330
43	1	17	456
47	1	18	2537
53	1	20	155518
59	1	22	195360
61	1	23	323371

TABLE 1..

Theorem 2.1. *For all primes* $p \geq 11$, $P(p,3) \leq (p-1)! - \lceil \frac{p}{3} \rceil + 2 \leq (p-1)! - 2$.

By strengthening Theorem 2.1, we managed to provide a new upper bound as follows:

Theorem 2.2. *For a prime number* n *and integer* $r \leq \frac{n}{6}$,

$$P(n,3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2 - 8rn + 20r^2}} \sqrt{\frac{(n-1)!}{n(n-r)!}}.$$

Let $k \in \mathbb{N}$ and $P(n,3) = (n-1)! - k$. Table 1. compares the values obtained for k from Theorems 1.1, 2.1 and 2.2 for prime numbers $31 \leq n \leq 61$. We note that for all prime numbers $11 \leq n \leq 31$, the upper bound obtained from Theorem 2.1 is better that the upper bound obtained from Theorem 2.2.

Also, we determine the exact value of $P(n,d)$ for all $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$ as follows:

Theorem 2.3. [2] $P(n,d) = 4$, *for all* $n \geq 6$ *and* $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$.

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