پنجاه و ششمین کنفرانس ریاضی ایران
56TH ANNUAL IRANIAN MATHEMATICS CONFERENCE

The Sequence Reconstruction of Permutations under Hamming Metric

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Abstract

Suppose that we are given N distinct permutations of the symmetric group S_n of degree n which are at most far from an unknown permutation χ by a positive integer $r > 1$ with respect to the Hamming distance. Under what conditions it is possible to determine χ uniquely? This question is answered positively by Levenshtein that there exists an integer $N(n,r)$ depending only on (n,r) such that if $N > N(n,r)$, then we are done. The new question is to determine exactly the least possible value of $N(n,r)$. The values of $N(n,r)$ is known for $r \in \{2,3,4\}$ by Wang, Fu and Konstantinova. Here we find $N(n,5)$.

Keywords: Rank modulation, Kendall τ-distance, permutation codes.
Mathematics Subject Classification (2020): 94B25; 94B65; 68P30.

1 Introduction

The sequence reconstruction problem, introduced by Levenshtein [3], involves reconstructing a transmitted sequence from multiple noisy copies received over distinct channels, each introducing at most r errors. This problem is motivated by applications in DNA storage [2], racetrack memories, and communication systems. In combinatorial terms, a key challenge is to determine the maximum intersection size $N(n,r)$ of two metric balls of radius r centered at distinct sequences. For permutations under the Hamming metric, this problem remains central due to the relevance of permutation codes in flash memories [1], power-line communications and DNA storage [4].

Main contributions of [5] resolves the sequence reconstruction problem for permutations under Hamming errors for small radii ($r \leq 4$) and provides asymptotic bounds for larger r.

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In particular it is proved that $N(n,2) = 3$ $(n \geq 3)$ [5, Theorem 4]; $N(n,3) = 4n - 6$ $(n \geq 3)$ [5, Theorem 5]; and $N(n,4) = 7n^2 - 31n + 36$ $(n \geq 4)$ [5, Theorem 6]. For example the latter implies that unique reconstruction requires $N(n,4)+1 = 7n^2-31n+37$ distinct permutations at distance ≤ 4.

2 Main Results

Theorem 2.1. *For any $n \geq 5$,*

$$N(n,5) = \frac{32}{3}n^3 - 89n^2 + \frac{739}{3}n - 220. \tag{1}$$

Proof. Here by using some elementary results about action of groups on sets, we find an algorithm with input r and output $N(n,r)$. Applying the algorithm we find the exact value of $N(n,5)$. $\square$

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