Improved Bounds on the Size of Permutation Codes Under Kendall τ -Metric

Farzad Parvaresh, Reza Sobhani, Alireza Abdollahi, Javad Bagherian, Fatemeh Jafari, and Maryam Khatami

***Abstract*—In order to overcome the challenges caused by flash memories and also to protect against errors related to reading information stored in DNA molecules in the shotgun sequencing method, the rank modulation method has been proposed. In the rank modulation framework, codewords are permutations. In this paper, we study the largest size $P(n,d)$ of permutation codes of length n, i.e., subsets of the set S_n of all permutations on $\{1,\dots,n\}$ with the minimum distance at least $d \in \{1,\dots,\binom{n}{2}\}$ under the Kendall τ-metric. By presenting an algorithm and two theorems, we improve the known lower and upper bounds for $P(n,d)$. In particular, we show that $P(n,d)=4$ for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$. Additionally, we prove that for any prime number n and integer $r \leq \frac{n}{6}$, $P(n,3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2-8rn+20r^2}}\sqrt{\frac{(n-1)!}{n(n-r)!}}$. This result greatly improves the upper bound of $P(n,3)$ for all primes $n \geq 37$.**

***Index Terms*—Rank modulation, Kendall τ-metric, permutation codes.**

I. Introduction

In ORDER to overcome the challenges caused by flash memories and also to protect against errors related to reading information stored in DNA molecules in the shotgun sequencing method, the rank modulation method has been proposed (see [17] and [19], respectively). In the rank modulation framework, codewords are permutations. Within this framework, permutation codes were extensively examined using three metrics: the Kendall τ-metric [1], [17], [25], [26], [30], the Ulam metric [11] and the ℓ_∞ metric [20], [24]. This study specifically concentrates on permutation codes under the Kendall τ-metric. Let n be a positive integer and let S_n denote the symmetric group on n letters, i.e., the set of all $n!$ permutations of the set $[n] := \{1,2,\dots,n\}$. Throughout this paper, for a permutation $\pi \in S_n$, we employ the vector notation of π as $[\pi(1),\pi(2),\dots,\pi(i),\pi(i+1),\dots,\pi(n)]$. A Permutation Code (PC) of length n represents a non-empty subset of S_n. In the context of a permutation $\pi \in S_n$, an adjacent transposition, denoted as $(i,i+1)$ for $1 \leq i \leq n-1$, transforms π into the permutation $[\pi(1),\pi(2),\dots,\pi(i+1),\pi(i),\dots,\pi(n)]$. The Kendall τ-distance between two permutations ρ and π in S_n, $d_K(\rho,\pi)$, is defined as the minimum number of adjacent transpositions required to express $\rho\pi^{-1}$ as their product. In the context of the Kendall τ-metric, a PC of length n with minimum distance d can correct up to $\lfloor\frac{d-1}{2}\rfloor$ errors induced by charge-constrained errors in flash memories, as in [17]. A central question in the theory of PCs is determining the value of $P(n,d)$, that is the size of the largest code in S_n with minimum Kendall τ-distance d, for $d \leq \binom{n}{2}$. The exact value of $P(n,d)$ is known for $d \in \{1,2\}$ and $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$ [7] and also for $n=5$ and for $n=6$ when $d \neq 3$ [30]. Furthermore, several researchers have presented bounds on $P(n,d)$ (see [1], [3], [7], [17], [25], [26], [30]).

In this paper, we present a theorem on the value of $P(n,d)$ as follows:

Theorem 1: $P(n,d)=4$, for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$.

Moreover, we achieved significant improvements on the lower bound of $P(n,d)$ when $n \in \{7,8\}$ by constructing new PCs from the subgroups of S_n (see Table II) and, in particular, we establish $P(7,12)=7$.

Utilizing sphere packing bound (see [17, Theorems 12 and 13]), $P(n,3) \leq (n-1)!$. In [9, Corollary 2.5 and Theorem 2.6] and [7, Corollary 2], it is proved that if $n > 4$ is a prime number or $4 \leq n \leq 10$, then $P(n,3) \leq (n-1)!-1$. In [1, Theorem 1.1], we improved the upper bound to $P(n,3) \leq (n-1)! - \lceil\frac{n}{3}\rceil + 2 \leq (n-1)!-2$ for all primes $n \geq 11$. Here we prove an additional upper bound on $P(n,3)$ as follows:

Theorem 2: For a prime number n and integer $r \leq \frac{n}{6}$,

$$P(n,3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2-8rn+20r^2}}\sqrt{\frac{(n-1)!}{n(n-r)!}}. \quad \text{(I.1)}$$

The upper bound for $P(n,d)$ derived from [1, Theorem 1.1] is superior to that from Theorem 2 for all prime numbers

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TABLE I
COMPARING THE UPPER BOUNDS OF $P(n, 3)$ OBTAINED FROM THEOREMS [1, THEOREM 1.1] AND THEOREM 2

n	[1, Theorem 1.1]	Theorem I.2
37	36! − 15	36! − 62
41	40! − 16	40! − 330
43	42! − 17	42! − 456
47	46! − 18	46! − 2537
53	52! − 20	52! − 155518
59	58! − 22	58! − 195360
61	60! − 23	60! − 323371

$11 \le n \le 31$. However, considering that every prime number greater than 5 can be written in the form of $6n+1$ or $6n+5$, the following corollary shows that Theorem 2 significantly improves the upper bound of $P(n, 3)$ for all prime numbers $n \ge 37$.

Corollary 1: Let $n \ge 37$ be a prime number, and define $r = \lfloor \frac{n}{6} \rfloor$. If $n \equiv 1 \pmod 6$, then Theorem 2 improves the known upper bound of $P(n, 3)$ by more than $1.61(5r+5)^{\frac{r-4}{2}} - \lfloor \frac{n}{3} \rfloor + 2$. Also, if $n \equiv 5 \pmod 6$, then the improvement is greater than $8.05(5r+9)^{\frac{r-4}{2}} - \lfloor \frac{n}{3} \rfloor + 2$.

In Table I, a comparison is made between the upper bounds of $P(n, 3)$ obtained from [1, Theorem 1.1] and Theorem 2 for prime numbers $37 \le n \le 61$.

The subsequent sections are organized as follows: In Section II, we provide the definitions and notations of PCs and summarize important results regarding bounds on $P(n, d)$. Section III presents a new table of values for lower bounds of $P(n, d)$ for $n \in \{5, 6, 7, 8\}$. In Section IV, we first prove Theorem 1, and subsequently, using a specific method, we determine the exact value of $P(7, 12)$. Finally, in Section V, we proceed to prove Theorem 2.

II. PRELIMINARIES

In this section, we first present some definitions and notations for PCs under Kendall τ-metric. Subsequently, we summarize key known results about the bounds used to determine the best known bounds on PCs under Kendall τ-metric in Table II. The composition of two permutations π and σ in S_n, denoted by $\sigma\pi$, is defined as $\sigma\pi(i) = \pi(\sigma(i))$ for all $i \in [n]$. The identity element of S_n is denoted by $\xi := [1, 2, \dots, n]$. For distinct elements $i, j \in [n]$, (i, j), which is called transposition, is the permutation obtained from exchanging i and j in ξ. For a permutation $\pi \in S_n$, let $I(\pi) := |\{(i, j) \in [n]^2 \,|\, i < j \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|$. In view of the parity of $I(\pi)$, π is called an even or odd permutation. For a set Q, $|Q|$ denotes the size of the set Q.

Let π and ρ be two permutations in S_n. There exists a well-known equivalent expression for $d_K(\rho, \pi)$ [17] as follows: $d_K(\rho, \pi) = |\{(i, j) \in [n]^2 \,|\, \rho^{-1}(i) < \rho^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|$. A PC $\mathcal{C}$ of length n is called an (n, d)-PC, if $d_K(\pi, \sigma) \ge d$ for all distinct elements $\pi, \sigma \in \mathcal{C}$. The largest size of an (n, d)-PC is denoted by $P(n, d)$ and a PC attaining this size is said to be optimal. It is known that $P(n, 1) = n!$, $P(n, 2) = \frac{n!}{2}$ and if $\frac{2}{3}\binom{n}{2} < d \le \binom{n}{2}$, then $P(n, d) = 2$ (see [7, Theorem 10]). In the following, we review some results that determine the best known bounds on $P(n, d)$. For a positive integer r and a permutation $\sigma \in S_n$, the ball of radius r which centered at σ in S_n under the Kendall τ-distance is denoted by $B_r(\sigma)$ defined by $B_r(\sigma) := \{\pi \in S_n \mid d_K(\sigma, \pi) \le r\}$. Since the Kendall τ-metric is right invariant (i.e., for every three permutations $\sigma, \pi, \rho \in S_n$ we have $d_K(\sigma, \pi) = d_K(\sigma\rho, \pi\rho)$ [7]), the size of a ball of radius r is independent of its center and we denote it by $B_K(r)$. The Gilbert-Varshamov bound and sphere-packing bound for PCs under Kendall τ-metric are as follows:

Proposition 1: [17, Theorems 12 and 13]

$$\frac{n!}{B_K(d-1)} \le P(n, d) \le \frac{n!}{B_K\left(\lfloor \frac{d-1}{2} \rfloor\right)}.$$

Let σ and τ be two permutations with $d_K(\sigma, \tau) = 1$. Then the double ball of radius r centered at σ and τ, denoted by $DB_r(\sigma, \tau)$, is defined by $DB_r(\sigma, \tau) := B_r(\sigma) \cup B_r(\tau)$. The size of $DB_r(\xi, [2, 1, 3, \dots, n])$ is denoted by $DB_{n,r}$. There are two useful results for bounds on $P(n, d)$, when d is even, as follows:

Proposition 2: For all n and $t \ge 1$,

(1) [7, Corollaries 5 & 6] $P(n, 2(t+1)) \le \dfrac{n!}{DB_{n,t}}$. Especially $P(n, 4) \le \dfrac{n!}{2(n-1)}$.

(2) [17, Theorem 21] $P(n, 2t) \ge \frac{1}{2} P(n, 2t-1)$.

The best known relation for the lower bound on $P(n, 3)$ is as follows:

Proposition 3: $P(n, 3) \ge \dfrac{n!}{2n-1}$ [17, p. 2116] and if $n-2$ is a prime power, then $P(n, 3) \ge \dfrac{n!}{2n-2}$ [3, Theorem 4.5].

Remark 1: By the part (ii) of Proposition 2 and Proposition 3, $P(n, 4) \ge \dfrac{n!}{2(2n-2)}$ if $n-2$ is a prime power and $P(n, 4) \ge \dfrac{n!}{2(2n-1)}$ otherwise.

There is an important improvement of the lower bound on $P(n, d)$, when $n-2$ is a prime power and $d > 4$ as follows:

Proposition 4: [25, Theorem 18] Let $m = ((n-2)^{t+1}-1)/(n-3)$, where $n-2$ is a prime power. Then $P(n, 2t+1) \ge \dfrac{n!}{(2t+1)m}$ and so $P(n, 2t+2) \ge \dfrac{n!}{2(2t+1)m}$.

If $\frac{1}{2}\binom{n}{2} < d \le \frac{2}{3}\binom{n}{2}$, then the following bound may turn out to be better than the sphere packing upper bound or part (1) of Proposition 2.

Proposition 5: [25, Theorem 23] If $P(n, 2t) \ge M$, then $2\binom{M}{2}t \le \binom{n}{2}\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil$ and if $P(n, 2t+1) \ge M$, then $(2t+2)\left(\binom{\lfloor \frac{M}{2} \rfloor}{2} + \binom{\lceil \frac{M}{2} \rceil}{2}\right) + (2t+1)\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil \le \binom{n}{2}\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil$.

III. CONSTRUCTING PERMUTATION CODES FROM COSETS OF SUBGROUPS

In this section, we devise an algorithm that, for integers n and d, attempts to determine the largest (n, d)-PC under the Kendall τ-metric, constructed by a subgroup and some of its left cosets among all subgroups of S_n. Employing GAP [10] through this algorithm allows us to discover new (n, d)-PCs under Kendall τ-metric, as detailed in Appendix, which

TABLE II
BEST KNOWN LOWER BOUND (LB) AND UPPER BOUND (UP) ON $P(n, d)$

$n \setminus d$		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17-18
5	LB=UB	20^i	12^i	6^i	5^i	2^c	2^c	2^c	2^c							
6	UB	116^b	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c				
	LB	102^i	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c				
7	UB	716^b	420^c	186^a	120^c	66^a	45^c	28^a	21^c	**10**	**7**	4^g	4^g	2^c	2^c	2^c
	LB	–	*336*	*126*	*84*	*42*	–	–	*13*	*8*	*7*	*4*	–	–	–	–
	LB	–	**315**	**126**	**84**	**42**	**28**	**15**	**12**	**8**	**7**	**4**	**4**	–	–	–
	OLB	588^c	294^f	110^d	55^c	34^d	17^f	14^d	7^a	2^a	2^a	2^a	2^a	2^c	2^c	2^c
8	UB	5039^c	2880^c	1152^a	720^c	363^a	242^c	141^a	99^c	64^a	47^c	32^a	25^c	10^g	8^g	4^g
	LB	*3752*	*2240*	*672*	*448*	*168*	*115*	*57*	*43*	*26*	*21*	*15*	*12*	*8*	–	–
	LB	**3696**	**2184**	**672**	**392**	**168**	**126**	**64**	**49**	**28**	**24**	**14**	**14**	**8**	**8**	**4**
	OLB	2688^h	1344^a	142^a	76^a	33^a	20^a	12^a	7^a	6^a	4^a	3^a	3^a	1^a	1^a	1^a

Key to the superscripts used in Table

superscript a	Sphere packing bound
superscript b	Sphere packing bound+[1, Theorem 3.5]
superscript c	[7, Corollary 5 or Theorems 10,12 or 13]
superscript d	Lower bounds from [17]
superscript f	[17, Theorem 21]
superscript g	[25, Theorem 23]
superscript h	$P(n,3) \geq \frac{n!}{2n-1}$ [17]
superscript i	[30, Table II]
an entry in bold	Tables V and VI and Theorem IV.6
an entry in italic	Lower bounds from [4]
blue entries	Best known lower bounds for $P(n,d)$, $n \in \{7,8\}$

improve the lower bounds of $P(n,d)$ for some values of d when $n \in \{7,8\}$. Subsequently, Table II is presented, illustrating the best-known bounds on $P(n,d)$ for $n \in \{5,6,7,8\}$. It is worth noting that recently, several improved lower bounds for $P(n,d)$ have been obtained in [4], using recursive techniques, automorphisms, and programs that combine randomness and greedy strategies. Notably, the bold and italic entries in the table represent results from the current paper and [4], respectively. Also the blue entries show the best known of lower bounds for $P(n,d)$, $n \in \{7,8\}$.

Before delving into the details of the algorithm, we first recall some relevant concepts. Suppose that H is a subgroup of a finite group G and $g \in G$. Then the sets $Hg := \{hg \mid h \in H\}$ and $gH := \{gh \mid h \in H\}$, where gh (or hg) refers to the group operation, are called a right coset of H and a left coset of H, respectively, with representative g. It is known that if $\mathbf{X}$ is the set of right (left) cosets of H in G, i.e., $\mathbf{X} := \{Hg \mid g \in G\}$ ($\mathbf{X} := \{gH \mid g \in G\}$), then $|\mathbf{X}| = |G|/|H|$ and $\mathbf{X}$ partitions G, i.e., $G = \cup_{X \in \mathbf{X}} X$ and $X \cap X' = \emptyset$ for all distinct elements X and X' of $\mathbf{X}$.

Description of Algorithm 1: Algorithm 1 consists of two primary functions:

- KDSET(M): Computes the minimum Kendall τ-distance between distinct elements of a subset M.
- KDELEM(M, g): Finds the minimum Kendall τ-distance between an element $g \in G$ and a set $M \subseteq G$.

The algorithm takes three input integers, n, d and $N_{\max}$. It initializes G and T as the symmetric group on the set $[n]$ and all non-trivial subgroups of G, respectively. Using GAP [10], access to all subgroups of G is possible for small values of n.

The algorithm initializes two empty lists: D and L_{best}. The list D will store all subgroups of G that form valid (n,d)-PCs. Note that since the Kendall τ-metric is right-invariant, for any subgroup $H \subseteq G$, we have $\min\{d_K(h,u) \mid h \neq u,\ h,u \in H\} = \min\{d_K(h,\xi) \mid h \in H \setminus \{\xi\}\}$. This property allows the algorithm to reduce the computational effort from $O(|H|^2)$ to $O(|H|)$ to compute the minimum Kendall τ-distance within subgroups, by using the function KDELEM($H \setminus \{\xi\}, \xi$) instead KDSET(H).

For each subgroup $H \in D$, the algorithm initializes a list L_H as a set of left coset representatives of H in G (i.e., $G := \dot{\bigcup}_{x \in L_H} xH$). Any representative $j \in L_H$ for which jH does not satisfy the minimum distance condition is removed. The objective is to identify the largest subset $S_H \subseteq L_H$ such that $\xi \in S_H$ and $\cup_{x \in S_H} xH$ forms an (n,d)-PC. To achieve this, the algorithm performs $N_{\max}$ iterations to find an optimal selection of cosets. Since the coset selection is done randomly, multiple iterations are necessary to improve the chances of finding a good solution. In our implementation, we set $N_{\max} = 1000$ to balance computational efficiency and solution quality. In some calculations for $n = 7$, we set $N_{\max}$ to values greater than 1000, but no improvement was observed. In each iteration, the following steps are executed:

- M_H is initialized as L_H, ensuring that each iteration begins with the full set of available representatives.
- Initializes a list M with the elements of H and an empty list S_H,
- While M_H is non-empty, a representative $j \in M_H$ is randomly selected and removed. If the minimum Kendall τ-distance between j and all elements of M is at least d, then j is added to S_H, and M is updated as $M \cup jH$.
- The largest selection S_H across all iterations is stored as S_H^{best}.

Note that if there exist $x, y \in T$ such that xH and yH are both (n,d)-PCs and $d_K(xh, y) \geq d$ for all $h \in H$, then by the right invariant property of the Kendall τ-metric, the union $xH \cup yH$ forms an (n,d)-PC. Thus, for each subgroup $H \in G$, the set M obtained in the previous step constitutes an (n,d)-PC.

Finally, once the best selection S_H^{best} is determined for each subgroup H, the pair $[H, S_H^{\text{best}}]$ is added to the output list L_{best}. By analysing the elements in L_{best}, the largest (n,d)-PC obtained through the algorithm can be identified.

During the coset selection process, cosets were chosen randomly rather than following an optimization criterion. A greedy approach was intentionally avoided due to its computational complexity, as evaluating all possible combinations at each step to identify the optimal choice significantly increases the running time.

It is worth noting that the construction of PCs using certain subgroups of symmetric groups and their cosets under various metrics has been previously studied (see [5], [13]). In [5], several PCs have been constructed using cosets of specific subgroups of S_n. These codes have been effective in improving the lower bounds of permutation codes under the Hamming metric, and some of them are still considered among the best-known PCs. As presented in Tables III and IV of the Appendix, for $n \in \{5, 6\}$, the (n, d)-PCs constructed using Algorithm 1 are either near-optimal or, in some cases, optimal. Also, as shown in Table II, the $(7, d)$-PCs, for $d \in \{12, 13, 14\}$ and the $(8, d)$-PCs, for $d \in \{16, 17\}$, constructed using Algorithm 1, are optimal. So, the PCs generated by Algorithm 1 appear to be good candidates for optimal or near-optimal PCs.

The complexity of Algorithm 1 is $O((n \times n!)^2 \times |T| \times N_{\max})$, where T, as defined in the algorithm, is the set of all non-trivial subgroups of S_n. The number of subgroups up to S_{18} can be found in [15].

IV. The Value of $P(n, d)$ for Certain Values of d

In this section, we initially present the proof for Theorem 1 and then determine the exact value of $P(7, 12)$. The proof of Theorem 1 relies on the following straightforward lemma.

Lemma 1: Let $n \geq 5$ be an integer. If $n \equiv 0, 2 \pmod 3$ ($n \equiv 1 \pmod 3$), then there exist 3 non-empty subsets with the same sum which partition $[n]$ ($[n] \setminus \{1\}$), respectively.

Proof: If n is $5, 6, 7, 8, 9$ and 10, respectively, then $\{\{5\}, \{1, 4\}, \{3, 2\}\}$, $\{\{6, 1\}, \{5, 2\}, \{3, 4\}\}$, $\{\{2, 7\}, \{3, 6\}, \{4, 5\}\}$, $\{\{8, 4\}, \{7, 3, 2\}, \{1, 5, 6\}\}$, $\{\{6, 5, 4\}, \{9, 1, 2, 3\}, \{8, 7\}\}$ and $\{\{10, 8\}, \{9, 2, 7\}, \{3, 4, 6, 5\}\}$ are the partitions of $[n]$ or $[n] \setminus \{1\}$ satisfying the lemma. Now, suppose that $n > 10$. Thus there exist $t > 0$ and $r \in \{5, 6, 7, 8, 9, 10\}$ such that $n = 6t + r$. Note that if $n \equiv 1 \pmod 3$, then $r \in \{7, 10\}$. Consider $t + 1$ subsets $\Theta_1, \ldots, \Theta_{t+1}$ of $[n]$ as follows:

$$\underbrace{1, \ldots, r}_{\Theta_1}, \underbrace{r+1, \ldots, r+6}_{\Theta_2}, \ldots, \underbrace{n-11, \ldots, n-6}_{\Theta_t}, \underbrace{n-5, \ldots, n}_{\Theta_{t+1}}.$$

Clearly, for all $2 \leq i \leq t + 1$, $\Theta_i = \{k_i + 1, k_i + 2, \ldots, k_i + 6\}$, where $k_i = r + (i - 2)6$. Hence, three sets $\Theta_{i1} := \{k_i + 1, k_i + 6\}$, $\Theta_{i2} := \{k_i + 2, k_i + 5\}$ and $\Theta_{i3} := \{k_i + 3, k_i + 4\}$ with the same sum $2k_i + 7$, partition the set Θ_i. Hence, $r \in \{5, 6, 7, 8, 9, 10\}$ implies that for each $1 \leq i \leq t + 1$, the set Θ_i is partitioned into three subsets Θ_{i1}, Θ_{i2} and Θ_{i3}, all of which have the same sum. Let $\Delta_j := \cup_{i=1}^{t+1} \Theta_{ij}$ for each $j \in \{1, 2, 3\}$. So Δ_1, Δ_2 and Δ_3 with the same sum partition $[n]$ or $[n] \setminus \{1\}$ if $n \equiv 0, 2 \pmod 3$ or $n \equiv 1 \pmod 3$, respectively. This completes the proof. $\square$

Proof of Theorem 1: It follows from [25, Theorem 23] that if $P(n, d) \geq 5$, then we must have $\binom{5}{2} d \leq 6 \times \binom{n}{2}$ and therefore $d \leq \frac{3}{5}\binom{n}{2}$. Thus, for all $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$, we have $P(n, d) \leq 4$.

Algorithm 1 Construction of (n, d)-PCs From Subgroups and Some of Their Cosets

Input: Integer values n, d, and $N_{\max}$ (number of random selection iterations)
Output: A list $[H, S_H]$ such that $\bigcup_{x \in \{\xi\} \cup S_H} xH$ is an (n, d)-PC with the best selection over $N_{\max}$ iterations
function KDSET(M) ▹ *Finds the minimum Kendall τ-distance between distinct elements of the set M*
  $S \leftarrow []$
  for all distinct elements i and j in M **do**
    add Kendall τ-distance between i and j to S
  end for
  return minimum of the list S
end function
function KDELEM(M, g) ▹ *Finds the minimum Kendall τ-distance between $g \in G$ and the elements of the set M*
  $S \leftarrow []$
  for all i in M **do**
    add Kendall τ-distance between g and i to S
  end for
  return minimum of the list S
end function
$G \leftarrow$ symmetric group on n letters
$T \leftarrow$ all non-trivial subgroups of G
$D \leftarrow []$ ▹ *List of subgroups satisfying distance constraints*
for all $Q \in T$ **do**
  if KDELEM$(Q \setminus \{\xi\}, \xi) \geq d$ **then**
    $D \leftarrow D \cup \{Q\}$
  end if
end for
$L_{\text{best}} \leftarrow []$ ▹ *Stores the best (n, d)-PC found*
for all $H \in D$ **do**
  $L_H \leftarrow$ left transversal set of H in G
  Remove all $j \in L_H$ for which KDSET$(jH) < d$
  $S_H^{\text{best}} \leftarrow []$ ▹ *Stores the best selection of coset representatives for H*
  for $k = 1$ to $N_{\max}$ **do**
    $S_H \leftarrow []$
    $M \leftarrow$ Elements of H ▹ *Stores the union of selected cosets*
    $M_H \leftarrow L_H$ ▹ *Refreshes available representatives for each iteration*
    while $M_H \neq \emptyset$ **do**
      Select $j \in M_H$ randomly and remove it from M_H
      if KDELEM$(M, j) \geq d$ **then**
        $S_H \leftarrow S_H \cup \{j\}$
        $M \leftarrow M \cup jH$
      end if
    end while
    if $|S_H| > |S_H^{\text{best}}|$ **then**
      $S_H^{\text{best}} \leftarrow S_H$
    end if
  end for
  $L_{\text{best}} \leftarrow L_{\text{best}} \cup \{[H, S_H^{\text{best}}]\}$
end for
return L_{best}

Since $P(n, d+1) \le P(n, d)$, it is enough to show that there exists an $\left(n, \lfloor 2/3\binom{n}{2}\rfloor\right)$-PC of size 4. Let $N := \sum_{i=1}^{n-1} i = \binom{n}{2}$. It follows from Lemma 1 that there exist pairwise distinct subsets Δ_1, Δ_2 and Δ_3 of $[n-1]$ or $[n-1] \setminus \{1\}$ such that if $n-1 \equiv 0, 2 \pmod 3$ or $n-1 \equiv 1 \pmod 3$, respectively, then $\sum_{j\in\Delta_i} j = \frac{N}{3}$ or $\sum_{j\in\Delta_i} j = \frac{N-1}{3}$, for all $i \in \{1,2,3\}$. Assume that for $n \ge 6$, the subsets Δ_1, Δ_2 and Δ_3 of $[n-1]$ are determined. Corresponding to each Δ_i, we introduce a permutation α_i as follows: let $r_i := |\Delta_i|$, $\Delta_i' := \{n-j \mid j \in \Delta_i\}$ and $\Phi_i := [n] \setminus \Delta_i'$. Suppose that $j_1 < j_2 < \cdots < j_{r_i}$ and $l_0 < l_1 < \cdots < l_{n-r_i-1}$ are all elements of Δ_i' and Φ_i, respectively. Let $\alpha_i \in S_n$ such that $\alpha_i(t) = j_t$ and $\alpha_i(n-s) = l_s$ for all $t \in \{1, \ldots, r_i\}$ and $s \in \{0, \ldots, n-r_i-1\}$. Let α_x and α_y be two distinct permutations corresponding to distinct subsets Δ_x and Δ_y, $x, y \in \{1,2,3\}$. In view of the definition of α_x, if $i < j$ are two elements of $[n]$, then $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ if and only if $i \in \Delta_x'$. So, since $\Delta_x' \cap \Delta_y' = \varnothing$, we have $(i,j) \in [n]^2$ satisfies $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ and $\alpha_y^{-1}(i) > \alpha_y^{-1}(j)$, if and only if $(i,j) \in A \cup B$, where $A := \{(i,j) \mid i < j, i \in \Delta_x'\}$ and $B := \{(i,j) \mid i > j, j \in \Delta_y'\}$. Hence

$$\begin{aligned} &d_K(\alpha_x, \alpha_y) \\ &= |\{(i,j) \mid \alpha_x^{-1}(i) < \alpha_x^{-1}(j) \wedge \alpha_y^{-1}(j) > \alpha_y^{-1}(i)\}| \\ &= |A \cup B| = |A| + |B|. \end{aligned}$$

Therefore, $d_K(\alpha_x, \alpha_y) = \sum_{i\in\Delta_x} i + \sum_{i\in\Delta_y} i$ and so $d_K(\alpha_x, \alpha_y)$ is equal to $\frac{2N}{3}$ if $n-1 \equiv 0, 2 \pmod 3$ and otherwise is equal to $\frac{2(N-1)}{3} = \lfloor \frac{2}{3}N \rfloor$. Also it is easy to see that

$$\begin{aligned} d_K(\xi, \alpha_x) &= |\{(i,j) \mid i < j \wedge \alpha_x^{-1}(i) > \alpha_x^{-1}(j)\}| \\ &= |\{(i,j) \mid i < j, i \in \Phi_x\}|, \end{aligned}$$

and therefore $d_K(\xi, \alpha_x)$ is equal to $N - \frac{N}{3} = \frac{2}{3}N$ if $n-1 \equiv 0, 2 \pmod 3$ and is equal to $N - \frac{N-1}{3} = \frac{2N+1}{3} > \lfloor \frac{2N}{3} \rfloor$ if $n-1 \equiv 1 \pmod 3$. Hence, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is an $\left(n, \lfloor \frac{2N}{3} \rfloor\right)$-PC of size 4. This completes the proof. □

Example 1: Let $n = 14$. Define the subsets $\Delta_1 := \{2,7,8,13\}$, $\Delta_2 := \{3,6,9,12\}$ and $\Delta_3 := \{4,5,10,11\}$. Each of these subsets has the same sum, 30, and together they form a partition of $\{2,3,\ldots,13\}$. Hence, by the proof of Theorem 1, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is an $(14, 60)$-PC, where

$$\begin{aligned} \alpha_1 &= [1,6,7,12,14,13,11,10,9,8,5,4,3,2], \\ \alpha_2 &= [2,5,8,11,14,13,12,10,9,7,6,4,3,1], \\ \alpha_3 &= [3,4,9,10,14,13,12,11,8,7,6,5,2,1]. \end{aligned}$$

Definition 1: A permutation code $\mathcal{C}$ is called equidistant (called EPC for short) under Kendall τ-metric whenever any two distinct permutations in $\mathcal{C}$ have the same Kendall τ-distance. The maximum size of the largest EPC of length n and Kendall τ-distance d is denoted by $EP(n,d)$. Also we denote by $P(n,d,m,d')$, the size of the largest PC with minimum Kendall τ-distance d in S_n such that it contains an EPC of size m and Kendall τ-distance d'.

The problem of determining bounds on EPCs under the Hamming metric dates back to the 1970s, beginning with a question of Bolton in [6]. Several studies, such as [12], [14], [27], and [29], have explored this topic due to its applications in powerline communications and balanced scheduling. For a brief overview of EPCs under the Hamming metric, the reader could refer to [8, Section VI.44.5]. However there is no dedicated study on EPCs under the Kendall τ-metric, and the only related work is [30, p. 3160], which studies the number of permutations at the same distance from the identity element. In the subsequent discussion, we leverage the notion of EPCs under the Kendall τ-metric to demonstrate that $P(7,12) = 7$.

Proposition 6:

1) For each $1 \le d \le \binom{n}{2}$ and $\sigma \in S_n$, there exists a permutation $\pi \in S_n$ such that $d_K(\sigma, \pi) = d$.
2) If d is odd, then $EP(n,d) = 2$.
3) $EP(n,d) = 2$, for all $2/3\binom{n}{2} < d \le \binom{n}{2}$.
4) $EP\left(n, 2/3\binom{n}{2}\right) = 4$.

Proof: Let $N = \binom{n}{2}$ and $\sigma^r := [\sigma(n), \ldots, \sigma(1)]$. Since $d_K(\sigma, \sigma^r) = N$, we can construct a sequence of N adjacent transpositions $\rho_1, \rho_2, \ldots, \rho_N$ in S_n such that $\sigma^r = \rho_N \cdots \rho_1 \sigma$. Now, for each $1 \le d \le \binom{n}{2}$, we set $\pi = \rho_d \cdots \rho_1 \sigma$. Therefore, $d_K(\sigma, \pi) = d$, which completes the proof of part (1).

Since the composition of two odd permutations (or two even permutations) is always even, the right-invariant property of the Kendall τ-metric implies that the Kendall τ-distance between two permutations of the same parity is even. Thus, part (2) follows from part (1). Similarly, part (3) follows from part (1) and [7, Theorem 10], while part (4) follows from the proof of Theorem 1. This completes the proof. □

Lemma 2: $EP(7,12) = 7$ and $P(7,11,6,12) = 7$.

Proof: Let $\mathcal{C}$ be an $(7,12)$−EPC under the Kendall τ− metric of maximum size. Without loss of generality, we may assume that $\xi \in \mathcal{C}$, as Kendall τ-metric is right invariant. Let $\mathcal{A} := \{\sigma \in S_n \mid d_K(\xi, \sigma) = 12\}$. Using GAP [10], $|\mathcal{A}| = 531$. It is sufficient to find the maximum EPC with Kendall τ-distance 12 in A. Let $\mathcal{A}_i$, $2 \le i \le 7$, be the set of all subsets of size i in $\mathcal{A}$ such that the Kendall τ-distance between any pair of distinct elements in any of them is equal to 12. Using GAP we find that $|\mathcal{A}_2| = 27697$, $|\mathcal{A}_3| = 172629$, $|\mathcal{A}_4| = 131777$, $|\mathcal{A}_5| = 10862$, $|\mathcal{A}_6| = 9$ and $|\mathcal{A}_7| = 0$. Hence, the largest EPC in $\mathcal{A}$ has size 6, and therefore $|\mathcal{C}| = 7$. Suppose that $\mathcal{C}$ is an $(7,11)$-PC such that contains an EPC $\bar{\mathcal{C}}$ of size 6 and the Kendall τ-distance 12. Without loss of generality we may assume that $\xi \in \bar{\mathcal{C}}$. According to the proof of the first part, $\bar{\mathcal{C}} \in \mathcal{M} := \{\{\xi\} \cup A \mid A \in \mathcal{A}_5\}$. So there are 10862 distinct cases for $\bar{\mathcal{C}}$. Let $\mathcal{B}_M := \{\sigma \in S_n \mid d_K(m, \sigma) \ge 11, \forall\, m \in M\}$, for all $M \in \mathcal{M}$. Using GAP, for all $M \in \mathcal{M}$, $0 \le |\mathcal{B}_M| \le 14$ and if $|\mathcal{B}_M| \ne 0$ and $b_1, b_2 \in \mathcal{B}_M$ then $d_K(b_1, b_2) < 11$. This completes the proof. □

Theorem 3: $P(7,12) = 7$ and $8 \le P(7,11) \le 10$.

Proof: Let $\mathcal{C}$ be an $(7,d)$−PC under the Kendall τ−metric and $\Sigma := \sum_{c_1,c_2\in\mathcal{C}} d_K(c_1,c_2)$. By the same argument as in the proof of [25, Theorem 23], we can see that $\Sigma \le \binom{n}{2}\lceil \frac{|\mathcal{C}|}{2} \rceil \lfloor \frac{|\mathcal{C}|}{2} \rfloor$. By Theorem 5, $P(7,12) \le 8$. As shown in Table II, $P(7,12) \ge 7$. Then it is sufficient to show that $P(7,12) \ne 8$. For a contradiction, assume that $\mathcal{C}$ is an $(7,12)$−PC of size 8. Thus, we must have $\Sigma \le 336$. On the other hand, since $EP(7,12) = 7$ and $|\mathcal{C}| = 8$, there exist c_1, c_2 in $\mathcal{C}$ such that $d_K(c_1,c_2) > 12$. Hence, $\Sigma \ge \binom{8}{2} \times 12 + 1 = 337$ which leads to a contradiction. So $P(7,12) = 7$.

As shown in Table II, $P(7,11) \ge 8$. Theorem 5 implies that $P(7,11) \le 12$. As a contradiction, assume that $\mathcal{C}$ is an

$(7,11)-$PC of size 12 or 11. Let $\mathcal{C}_1 := \mathcal{C} \cap A_n$ and $\mathcal{C}_2 := \mathcal{C}\backslash\mathcal{C}_1$, where A_n denotes the set of all even permutations in S_n. Without loss of generality, we may assume that $|\mathcal{C}_1| \geq |\mathcal{C}_2|$. Since the Kendall τ-distance between two permutations of the same parity is even, if c_1 and c_2 are two distinct elements in $\mathcal{C}_1$ or $\mathcal{C}_2$, then $d_K(c_1, c_2) \geq 12$. So

$$\binom{|\mathcal{C}_1|}{2} \times 12 + \binom{|\mathcal{C}_2|}{2} \times 12 + 11 \times |\mathcal{C}_1| \times |\mathcal{C}_2| \leq \Sigma.$$

If $|\mathcal{C}| = 12$ and 11, then $\Sigma \leq 756$ and 630, respectively. Hence it can be seen that if $|\mathcal{C}| = 12$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ must be two $(7,12)-$EPC of sizes 6 and also if $|\mathcal{C}| = 11$, then $\mathcal{C}_1$ and $\mathcal{C}_2$ must be two $(7,12)-$EPC of sizes 6 and 5, respectively. Therefore if $|\mathcal{C}| \in \{11, 12\}$, then $\mathcal{C}$ is an $(7,11)-$PC such that contains an $(7,12)-$EPC of size 6 which contradicts Lemma 2, This completes the proof. □

V. New Upper Bound of $P(n,3)$

In [1], we formulate an integer programming problem that depends on the choice of a non-trivial subgroup H of S_n. The optimal value of the objective function obtained from this formulation provides an upper bound for $P(n,3)$ (see [1, Theorem 2.14]). To improve the upper bound for $P(n,3)$, we consider this integer programming problem for a fixed subgroup H chosen from the collection of Young subgroups, which are well-studied subgroups of S_n (see [18]). The definition of a Young subgroup is provided next.

Definition 2: A integer partition λ of n (with length m) refers to an ordered m-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $n = \sum_{i=1}^{m} \lambda_i$. Let λ be a partition of n, and let $\Delta = (\Delta_1, \ldots, \Delta_m)$ be an m-tuple of non-empty subsets of $[n]$ that partitions $[n]$, with each Δ_i having $|\Delta_i| = \lambda_i$ for $i = 1, \ldots, m$. We define a Young subgroup S_Δ of S_n as follows:

$$S_\Delta = S_{\Delta_1} \times S_{\Delta_2} \times \cdots \times S_{\Delta_m},$$

where S_{Δ_i} is the symmetric group on the set Δ_i for each $i = 1, \ldots, m$.

In [1, Theorem 1.1], by concentrating on the integer programming problem associated with a Young subgroup corresponding to the partition $(n-1,1)$ of n, we improve the upper bound for $P(n,3)$ from $(n-1)!-1$ to $(n-1)!-\lceil \frac{n}{3} \rceil+2$ for all primes $n \geq 11$. Note that for the partition $\lambda := (1,1,\ldots,1)$ of n, the integer programming problem provides the exact value of $P(n,3)$. In this section, through the application of a new method to examine the integer programming problem related to the Young subgroup corresponding to the partition $\lambda := (n-r, \underbrace{1,\ldots,1}_{r})$ of n, where $r \leq \frac{n}{6}$, we improve the upper bound for $P(n,3)$ as outlined in Theorem 2 for all primes $n \geq 37$. In this section, we will adhere to the definitions and notations provided in [1]. Specifically, we utilize the following definition:

Definition 3: (see [1, Definition 2.10 and Remark 2.12]) Let H be a subgroup of S_n, and let $\{\sigma_1, \ldots, \sigma_m\}$ be a set of right transversal elements of H in G, where $m = \frac{n!}{|H|}$. That is, let $X = \{H\sigma_1, \ldots, H\sigma_m\}$ is the set of right cosets of H in S_n. We fix an ordering on X such that $H\sigma_i < H\sigma_j$ whenever $i < j$. Then $\rho_X^{S_n}$ is a map from S_n to $GL_m(\mathbb{Z})$ (the group of all invertible $m \times m$ matrices with integer entries), defined by $\rho \to P_\rho$, where P_ρ is the $m \times m$ matrix whose (i,j)-entry is 1 if $H\sigma_i\rho = H\sigma_j$ and 0 otherwise. Moreover, if $Y \subseteq S_n$, then $\widehat{Y^{\rho_X^{S_n}}}$ represents the element $\sum_{y \in Y} y^{\rho_X^{S_n}} = \sum_{y \in Y} P_y$ in $Mat_m(\mathbb{Z})$, the set of all $m \times m$ matrices over $\mathbb{Z}$.

Example 2: Consider the partition $\{\Delta_1, \Delta_2\}$ of the set $\{1,2,3\}$, where $\Delta_1 = \{1,2\}$ and $\Delta_2 = \{3\}$. Then $H = \{\sigma_1\sigma_2 \,|\, \sigma_1 \in S_{\{1,2\}}, \sigma_2 \in S_{\{3\}}\}$ is a Young subgroup corresponding to the partition $\lambda = (2,1)$ of $n = 3$. Clearly, H is the subgroup generated by the transposition $(1,2)$ in S_3 and $X = \{H, H(1,3), H(2,3)\}$ is the set of right cosets of H in S_3. We fix the order $H < H(23) < H(13)$. Then, P_ξ is the identity matrix of order 3, $P_{(1,2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $P_{(2,3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Also if $T = \{\xi, (1,2), (2,3)\}$, then $\widehat{T^{\rho_X^{S_n}}} = P_\xi + P_{(1,2)} + P_{(2,3)}$ that is the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

To prove Theorem 2, we need the following lemma.

Lemma 3: Let H be the Young subgroup of S_n corresponding to the partition $\lambda := (n-r, \underbrace{1,\ldots,1}_{r})$ and X be the set of right cosets of H in S_n. If $S = \{(i,i+1) \,|\, 1 \leq i \leq n-1\}$ and $T := S \cup \{\xi\}$, then $\widehat{T^{\rho_X^{S_n}}}$ is a symmetric matrix $A = (a_{ij})_{\ell \times \ell}$, where $\ell = \frac{n!}{(n-r)!}$, with the following properties:

1) $a_{ii} \geq n - 2r$ for all $i \in [\ell]$.
2) $a_{ij} \in \{0,1\}$ for all $i \neq j \in [\ell]$.
3) $\sum_{j=1}^{\ell} a_{ij} = n$ for all $i \in [\ell]$.

Proof: In view of [1, Remark 3.2], without loss of generality, we may assume that λ is the partition $\{[n-r], \{n-r+1\}, \{n-r+2\}, \ldots, \{n\}\}$ of $[n]$, and therefore, $H \cong S_{n-r}$. Let $\mathcal{F}$ be the set $\{(f_1, f_2, \ldots, f_r) \in [n]^r \,|\, \forall\, i \neq j, f_i \neq f_j\}$. Corresponding to each ordered r-tuple $F = (f_1, \ldots, f_r) \in \mathcal{F}$, we let $S_n^F := \{\sigma \in S_n \,|\, \sigma(n-r+1) = f_1, \sigma(n-r+2) = f_2, \ldots, \sigma(n) = f_r\}$. It is easy to see that $S_n^F = H\sigma$ for each $\sigma \in S_n^F$. Hence, S_n^F is a right coset of H in S_n. Furthermore, if F and $\bar{F}$ are two distinct elements of $\mathcal{F}$, then $S_n^F \cap S_n^{\bar{F}} = \varnothing$. Since $|\mathcal{F}| = \ell$, $X = \{S_n^F \,|\, F \in \mathcal{F}\}$ is the set of all right cosets of H in S_n. Suppose that $F_1, F_2, \ldots, F_\ell$ are all ordered r-tuples in $\mathcal{F}$. Fix the ordering of X such that $S_n^{F_i} < S_n^{F_j}$ if $i < j$, for all $i, j \in [\ell]$. In view of Definition 3, the (i,j) entry of $\widehat{T^{\rho_X^{S_n}}}$ is equal to $|\mathcal{O}_{ij}|$, where $\mathcal{O}_{ij} := \{t \in T \,|\, S_n^{F_i} t = S_n^{F_j}\}$. Since $\mathcal{O}_{ij} = \mathcal{O}_{ji}$ for all $i, j \in [\ell]$, A is a symmetric matrix.

Let $(i, i+1) \in T$, and let $F = (f_1, \ldots, f_r)$ and $\bar{F} = (\bar{f}_1, \ldots, \bar{f}_r)$ be two distinct elements of $\mathcal{F}$. A sufficient condition for $S_n^F(i,i+1) = S_n^F$ is $\{i, i+1\} \cap \{f_1, \ldots, f_r\} = \varnothing$. So $a_{ss} \geq n - 2r$ for all $s \in [\ell]$.

As a contradiction assume that there exists $(j, j+1) \in T \setminus \{(i,i+1)\}$ such that $S_n^F(i,i+1) = S_n^{\bar{F}}$ and $S_n^F(j,j+1) = S_n^{\bar{F}}$. Since $F \neq \bar{F}$, we have $P_1 := \{f_1, \ldots, f_r\} \cap \{i, i+1\} \neq \varnothing$ and also $\{f_1, \ldots, f_r\} \cap \{j, j+1\} \neq \varnothing$. Suppose that $i \in P_1$ and $f_m = i$ for some $m \in [r]$. Then for all $\sigma \in S_n^F$, $\big(\sigma(i,i+1)\big)(n-r+m) = i+1$ and $\big(\sigma(j,j+1)\big)(n-r+m)$ is equal to j if $i = j+1$, and is equal to i if $\{i, i+1\} \cap \{j, j+1\} = \varnothing$. So $S_n^F(i,i+1) \neq S_n^F(j,j+1)$, which is a contradiction.

Now, suppose that $i+1 \in P_1$ and $f_d = i+1$ for some $d \in [r]$. Then, by the same argument, it can be seen that $\big(\sigma(i,i+1)\big)(n-r+d) \neq \big(\sigma(j,j+1)\big)(n-r+d)$, for all $\sigma \in S_n^F$. Hence, $S_n^F(i,i+1) \neq S_n^F(j,j+1)$, which leads to a contradiction. Therefore, $a_{ij} \in \{0,1\}$ for all $i \neq j \in [\ell]$. Note that for each $x \in [\ell]$, since $\cup_{y=1}^{\ell} \mathcal{O}_{xy} = T$ and $\mathcal{O}_{xy} \cap \mathcal{O}_{xy'} = \varnothing$ for all $y \neq y' \in [\ell]$, we have $\sum_{j=1}^{\ell} a_{ij} = n$ for all $i \in [\ell]$. This completes the proof. □

Here, we provide some notations used in the proof of Theorem 2. The transpose of a matrix or vector is denoted by $(\cdot)^t$. The inner product of two vectors $\mathbf{x} = (x_1,\ldots,x_n)^t$ and $\mathbf{y} = (y_1,\ldots,y_n)^t$ in $\mathbb{R}^n$ is defined as $\langle \mathbf{x},\mathbf{y}\rangle := \mathbf{x}^t\mathbf{y} = \sum_{i=1}^n x_i y_i$, the notation $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$ denotes the 2-norm of vector $\mathbf{x}$ and the notation $\|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i|$ denotes the 1-norm of vector $\mathbf{x}$, where $|a|$ denotes the absolute value of real number a. In the following, we state a definition and a remark that will play an important role in the proof of Theorem 4.

Definition 4: [23] A polyhedral cone is a subset $\mathcal{C} \subset \mathbb{R}^n$ of the form $\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} \leq \mathbf{0}\}$, for a matrix $A \in R^{m\times n}$ and column vector $\mathbf{0}$ of order $n \times 1$ whose entries are equal to 0.

Remark 2: Let $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \,|\, A\mathbf{x} \leq \mathbf{0}\}$ be a polyhedral cone for a non-singular matrix $A \in \mathbb{R}^{n\times n}$. In view of [23], the vector $\mathbf{d} \in \mathbb{R}^n$ is called an extreme ray of $\mathcal{C}$, if there exists $1 \leq i \leq n$ such that $A_i\mathbf{d} = \mathbf{0}$ and $a_i\mathbf{d} \leq 0$, where a_i denotes the i-th row of the matrix A and A_i is the submatrix of A obtained by removing a_i. We say that two extreme rays $\mathbf{d}$ and $\mathbf{d}'$ of $\mathcal{C}$ are equivalent, and denote it by $\mathbf{d} \sim \mathbf{d}'$, if one is a positive multiple of the other. In view of [23, p. 101-105], the number of equivalence classes of extreme rays in $\mathcal{C}$ is finite. Also according to [23, p. 105], if $\{\mathbf{w}_1,\ldots,\mathbf{w}_s\}$ is a complete set of representatives of all equivalence classes of extreme rays in $\mathcal{C}$, then $\mathcal{C} = \left\{\sum_{i=1}^s \lambda_i \mathbf{w}_i \,|\, \lambda_i \geq 0\right\}$.

Theorem 4: Let r and n be integers such that $r \leq \frac{n}{6}$ and $n \nmid (n-r)!$. Then

$$P(n,3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2-8rn+20r^2}}\sqrt{\frac{(n-1)!}{n(n-r)!}}.$$

Proof: Let C be a PC in S_n with minimum Kendall τ-distance 3. Let H be the Young subgroup of S_n corresponding to the partition $\lambda := (n-r, \underbrace{1,\ldots,1}_{r})$ and Y be the set of right cosets of H in S_n. If $S = \{(i,i+1) \,|\, 1 \leq i \leq n-1\}$ and $T := S \cup \{\xi\}$, then by Lemma 3, $T^{\widehat{\rho_Y^{S_n}}}$ is a matrix $A = (a_{ij})_{\ell\times\ell}$, $\ell = \frac{n!}{(n-r)!}$, with properties specified in Lemma 3. Theorem [1, 2.14] implies that the optimal value of the objective function of the following integer programming problem gives an upper bound on $|C|$

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=1}^{\ell} x_i, \\ \text{subject to} \quad & A(x_1,\ldots,x_\ell)^t \leq |H|\mathbf{1} = (n-r)!\mathbf{1}, \\ & x_i \in \mathbb{Z},\ x_i \geq 0,\ i \in \{1,\ldots,\ell\}, \end{aligned}$$

where $\mathbf{1}$ is a column vector of order $\ell \times 1$ whose entries are equal to 1. Let $\boldsymbol{\alpha}$ be a feasible solution for the above linear inequality system that achieves the optimum of the objective function and $\boldsymbol{\beta} := \frac{(n-r)!}{n}\mathbf{1}$. It follows from the part (3) of Lemma 3 that the sum of every row in A is equal to n and so $A\boldsymbol{\beta} = (n-r)!\mathbf{1}$. Since $n \nmid (n-r)!$ we have $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. It is clear that $\sum_{i=1}^{\ell} \alpha_i \leq (n-1)!$, where α_i denotes the i-th entry of $\boldsymbol{\alpha}$, and suppose that $\sum_{i=1}^{\ell} \alpha_i = (n-1)! - k$ for a non-negative integer k. Consider two vectors $\overrightarrow{\boldsymbol{\beta\alpha}} := \boldsymbol{\alpha} - \boldsymbol{\beta}$ and $-\mathbf{1}$. We let

$$\begin{aligned} \mu &:= \frac{\left\langle -\mathbf{1}, \overrightarrow{\boldsymbol{\beta\alpha}}\right\rangle}{\|-\mathbf{1}\| \left\|\overrightarrow{\boldsymbol{\beta\alpha}}\right\|} = \frac{\langle -\mathbf{1}, \boldsymbol{\alpha}-\boldsymbol{\beta}\rangle}{\|-\mathbf{1}\|\,\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|} = \frac{\langle -\mathbf{1}, \boldsymbol{\alpha}\rangle + \langle -\mathbf{1}, -\boldsymbol{\beta}\rangle}{\|-\mathbf{1}\|\,\|\boldsymbol{\alpha}-\boldsymbol{\beta}\|} \\ &= \frac{\ell\frac{(n-r)!}{n} - \sum_{i=1}^{\ell}\alpha_i}{\sqrt{\ell}\sqrt{\sum_{i=1}^{\ell}(\alpha_i-\beta_i)^2}} = \frac{k}{\sqrt{\ell}\sqrt{\sum_{i=1}^{\ell}(\alpha_i-\beta_i)^2}}, \end{aligned}$$

where β_i denotes the i-th entry of $\boldsymbol{\beta}$. Since for each $i \in [\ell]$, α_i is an integer, we have $|\alpha_i - \beta_i| \geq \frac{1}{n}$. Hence,

$$k \geq \mu\sqrt{\ell}\sqrt{\frac{\ell}{n^2}} = \mu\frac{\ell}{n} = \frac{(n-1)!}{(n-r)!}\mu. \tag{V.1}$$

Let $\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^\ell \,|\, A\mathbf{x} \leq (n-r)!\mathbf{1}\} = \{\mathbf{x} \in \mathbb{R}^\ell \,|\, A(\mathbf{x}-\boldsymbol{\beta}) \leq \mathbf{0}\}$. In view of Definition 4, $\mathcal{C}$ is a polyhedral cone. Note that since $r \leq \frac{n}{6}$, Lemma 3 implies that $A = (a_{ij})_{\ell\times\ell}$ is a matrix such that $a_{ii} > \sum_{i\neq j=1}^{\ell} a_{ij}$ for all $1 \leq i \leq \ell$. Therefore Levy-Desplanques Theorem [16, p. 125] implies A is a non-singular matrix. Also, since $\lambda_0\mathbf{u} + (1-\lambda_0)\mathbf{v} \in \mathcal{C}$ for all $\mathbf{u},\mathbf{v} \in \mathcal{C}$ and $\lambda_0 \in [0,1]$, $\mathcal{C}$ is a convex set. It is clear that $\boldsymbol{\beta},\boldsymbol{\alpha} \in \mathcal{C}$ and so the vector $\overrightarrow{\boldsymbol{\beta\alpha}}$ belongs to $\mathcal{C}$. Suppose that $\{\mathbf{w}_1,\ldots,\mathbf{w}_s\}$ is a complete set of representatives of all equivalence classes of extreme rays in $\mathcal{C}$ such that $\|\mathbf{w}_i\| = 1$ for all $1 \leq i \leq s$. Since $\overrightarrow{\boldsymbol{\beta\alpha}} \in \mathcal{C}$, it follows from Remark 2 that there exist non-negative real numbers $\lambda_1,\ldots,\lambda_s$ such that $\overrightarrow{\boldsymbol{\beta\alpha}} = \sum_{i=1}^s \lambda_i\mathbf{w}_i$. Then

$$\mu = \frac{\left\langle -\mathbf{1}, \overrightarrow{\boldsymbol{\beta\alpha}}\right\rangle}{\|\mathbf{1}\| \left\|\overrightarrow{\boldsymbol{\beta\alpha}}\right\|} = \frac{\left\langle -\mathbf{1}, \sum_{i=1}^s \lambda_i\mathbf{w}_i\right\rangle}{\|-\mathbf{1}\| \left\|\sum_{i=1}^s \lambda_i\mathbf{w}_i\right\|}.$$

Since $\left\|\sum_{i=1}^s \lambda_i\mathbf{w}_i\right\| \leq \sum_{i=1}^s \lambda_i\|\mathbf{w}_i\|$,

$$\mu \geq \frac{\sum_{i=1}^s \lambda_i\langle -\mathbf{1}, \mathbf{w}_i\rangle}{\|-\mathbf{1}\|\left(\sum_{i=1}^s \lambda_i\|\mathbf{w}_i\|\right)},$$

and since $\|\mathbf{w}_i\| = 1$ for all $1 \leq i \leq s$,

$$\begin{aligned} \mu \geq \sum_{i=1}^s \frac{\lambda_i\langle -\mathbf{1}, \mathbf{w}_i\rangle}{\left(\sum_{j=1}^s \lambda_j\right)\|-\mathbf{1}\|} &= \sum_{i=1}^s \frac{\lambda_i}{\sum_{j=1}^s \lambda_j}\frac{\langle -\mathbf{1}, \mathbf{w}_i\rangle}{\|-\mathbf{1}\|} \\ &\geq \sum_{i=1}^s \frac{\lambda_i}{\sum_{j=1}^s \lambda_j}\mu_0 = \mu_0, \end{aligned} \tag{V.2}$$

where $\mu_0 := \min\left\{\frac{\langle -\mathbf{1}, \mathbf{w}_i\rangle}{\|-\mathbf{1}\|} \,\Big|\, 1 \leq i \leq s\right\}$. Suppose that $\mu_0 = \frac{\langle -\mathbf{1}, \mathbf{w}_b\rangle}{\|-\mathbf{1}\|}$ for some $1 \leq b \leq s$. Hence, it follows from Remark 2 that there exists $i \in [n]$ such that $A_i\mathbf{w}_b = \mathbf{0}$ and $a_i\mathbf{w}_b \leq 0$, where a_i is the i-th row of the matrix A and A_i is the matrix obtained by removing a_i of the matrix A. According to the properties of the matrix A, without loss of generality, we may assume that $i = \ell$. Suppose that $\boldsymbol{\rho}$ is the ℓ-th column of A_ℓ and J is the $(\ell-1)\times(\ell-1)$ matrix obtained by removing the column $\boldsymbol{\rho}$ of the matrix A_ℓ. Levy-Desplanques Theorem implies J is a non-singular matrix.

Hence, $A_\ell(x_1,\dots,x_\ell)^t = J(x_1,\dots,x_{\ell-1})^t + \boldsymbol{\rho} x_\ell = \mathbf{0}$ implies $(x_1,\dots,x_{\ell-1})^t = -J^{-1}\boldsymbol{\rho} x_\ell$.

In the sequel, we show that $a_\ell\binom{J^{-1}\boldsymbol{\rho}}{-1} \le 0$ and therefore by placing $x_\ell = -1$ we have $\binom{-J^{-1}\boldsymbol{\rho}x_\ell}{x_\ell} \sim \mathbf{w}_r$. It follows from [28, Theorem 1] and Lemma 3 that if $\Delta := \min\left\{|J_{ii}| - \sum_{j=1,j\neq i}^{\ell-1}|J_{ij}| \,|\, 1 \le i \le \ell-1\right\}$, then $\left\|J^{-1}\right\|_\infty := \max\left\{\sum_{j=1}^{\ell-1}|(J^{-1})_{ij}| \,|\, 1 \le i \le \ell-1\right\} \le \frac{1}{\Delta}$. So Lemma 3 implies $\left\|J^{-1}\right\|_\infty \le \frac{1}{n-4r}$. Also if $|A| := (|a_{ij}|)_{n\times n}$ for a matrix $A = (a_{ij})_{n\times n}$, then we have

$$\left\|J^{-1}\boldsymbol{\rho}\right\|_1 = \mathrm{tr}(|J^{-1}\boldsymbol{\rho}|\mathbf{1}^t) \le \mathrm{tr}(|J^{-1}|\boldsymbol{\rho}\mathbf{1}^t).$$

Since the inverse of a symmetric matrix is a symmetric matrix, J^{-1} is a symmetric matrix. Let ρ_i denote the i-th entry of $\boldsymbol{\rho}$. It follows from Lemma 3 that $\rho_i \in \{0,1\}$ for all $1 \le i \le \ell-1$ and if $\tau := \{i \in [\ell-1] \,|\, \rho_i = 1\}$, then the size of τ is at most $2r$. Then we have

$$\begin{aligned}\mathrm{tr}(|J^{-1}|\boldsymbol{\rho}\mathbf{1}^t) &= \sum_{i=1}^{\ell-1}\sum_{j\in\tau}\left|(J^{-1})_{ij}\right| = \sum_{j\in\tau}\sum_{i=1}^{\ell-1}\left|(J^{-1})_{ij}\right| \\ &= \sum_{j\in\tau}\sum_{i=1}^{\ell-1}\left|(J^{-1})_{ji}\right| \le \sum_{j\in\tau}\left\|J^{-1}\right\|_\infty \\ &\le 2r\left\|J^{-1}\right\|_\infty,\end{aligned}$$

and therefore,

$$\left\|J^{-1}\boldsymbol{\rho}\right\|_1 \le \frac{2r}{n-4r}. \quad \text{(V.3)}$$

So, parts (1) and (2) of Lemma 3 and $r \le \dfrac{n}{6}$ imply that

$$a_\ell(J^{-1}\boldsymbol{\rho}, -1)^t \le \left\|J^{-1}\boldsymbol{\rho}\right\|_1 - (n-2r) \le 0$$

and so $\binom{J^{-1}\boldsymbol{\rho}}{-1} \sim \mathbf{w}_b$. Hence,

$$\begin{aligned}\mu_0 &= \frac{\left\langle -\mathbf{1}, \binom{J^{-1}\boldsymbol{\rho}}{-1}\right\rangle}{\|\mathbf{1}\|\left\|\binom{J^{-1}\boldsymbol{\rho}}{-1}\right\|} = \frac{1-\left\langle \mathbf{1}, J^{-1}\boldsymbol{\rho}\right\rangle}{\sqrt{\ell}\sqrt{1+\left\|J^{-1}\boldsymbol{\rho}\right\|^2}} \\ &\ge \frac{1-\left\|J^{-1}\boldsymbol{\rho}\right\|_1}{\sqrt{\ell}\sqrt{1+\left\|J^{-1}\boldsymbol{\rho}\right\|_1^2}}. \quad \text{(V.4)}\end{aligned}$$

Hence, relations (V.3) and (V.4) imply

$$\mu_0 \ge \frac{n-6r}{\sqrt{\ell}\sqrt{n^2-8rn+20r^2}}, \quad \text{(V.5)}$$

and therefore the result follows from relations (V.1), (V.2) and (V.5). This completes the proof. □

Proposition 7: For integers $n \ge 10$ and $r \le \dfrac{n}{2}$, if $n \nmid (n-r)!$, then n is a prime number.

Proof: As a contradiction, assume that n is not prime. Hence there exist $n_1, n_2 \in \mathbb{N}\setminus\{1\}$ such that $n = n_1 n_2$. Suppose first that $n_1 \neq n_2$ and $n_1 < n_2$. If $n_2 \le n-r$, then $n|(n-r)!$ that is a contradiction. So $n_2 > n-r$. Since $r \le \frac{n}{2}$,

$$\frac{n}{2} \le n-r < n_2 = \frac{n}{n_1},$$

and therefore $n_1 < 2$ that is a contradiction. Now, suppose that $n = n_1^2$. Since $n \nmid (n-r)!$, $n-r < 2n_1$ and so

$$\frac{n_1^2}{2} = \frac{n}{2} \le n-r < 2n_1,$$

TABLE III

$(5,d)$–PCs OBTAINED FROM ALGORITHM 1

d	$\lvert C_5^d\rvert$	Generators of H	$\lvert H\rvert$	Elements of S_H	$\lvert\mathcal{C}\rvert$	$P(5,d)$
3	67	[3, 2, 1, 5, 4] [2, 4, 1, 3, 5]	20	–	20	20
4	47	[4, 5, 3, 1, 2] [1, 3, 5, 4, 2]	6	[2, 5, 1, 3, 4]	12	12
5	23	[2, 4, 5, 3, 1]	5	–	5	6
6	14	[2, 4, 5, 3, 1]	5	–	5	5

TABLE IV

$(6,d)$–PCs OBTAINED FROM ALGORITHM 1

d	Generators of H	$\lvert H\rvert$	Elements of S_H	$\lvert\mathcal{C}\rvert$	$P(6,d)$
3	[5, 3, 1, 4, 2, 6] [3, 5, 6, 2, 4, 1]	24	[1, 3, 2, 5, 6, 4] [1, 3, 4, 2, 6, 5] [1, 3, 4, 5, 6, 2]	96	≥ 102
4	[2, 6, 5, 3, 4, 1]	3	[1, 2, 6, 3, 5, 4] [1, 2, 4, 6, 5, 3] [1, 5, 2, 3, 6, 4] [1, 3, 6, 2, 4, 5] [1, 4, 3, 2, 6, 5] [1, 4, 5, 2, 3, 6] [1, 3, 5, 4, 2, 6] [1, 5, 6, 4, 2, 3] [1, 6, 3, 5, 4, 2] [3, 1, 2, 5, 6, 4] [5, 1, 2, 4, 6, 3] [5, 1, 6, 3, 2, 4] [4, 1, 6, 5, 2, 3] [3, 1, 4, 6, 5, 2] [4, 1, 3, 5, 6, 2] [5, 1, 4, 3, 6, 2] [3, 4, 1, 2, 5, 6] [3, 5, 1, 6, 4, 2] [5, 3, 4, 1, 2, 6]	60	64
5	[1, 4, 5, 3, 6, 2]	5	[6, 2, 3, 4, 5, 1] [5, 1, 6, 3, 4, 2] [5, 2, 1, 3, 4, 6] [4, 2, 6, 3, 1, 5]	25	26
6	[4, 2, 3, 1, 6, 5] [3, 2, 4, 5, 6, 1]	10	[4, 1, 5, 3, 2, 6]	20	20
7	[4, 3, 1, 6, 5, 2]	5	[3, 5, 2, 4, 1, 6]	10	11
8	[4, 2, 6, 1, 5, 3] [5, 6, 2, 1, 4, 3]	6	–	6	7
9	[1, 6, 5, 4, 3, 2]	2	[5, 2, 4, 6, 3, 1]	4	4
10	[1, 6, 5, 4, 3, 2]	2	[5, 2, 4, 6, 3, 1]	4	4

and therefore $n_1 < 4$ that is a contradiction. This completes the proof. □

Remark 3: In view of Proposition 7, the only numbers that satisfy the assumptions of Theorem 4 are prime numbers. Thus, Theorem 4 is interchangeable with Theorem 2.

Proof of Corollary 1: First, consider the case where $n \equiv 1 \pmod 6$. In this case, we have $r = \frac{n-1}{6}$. Hence, it follows from inequality (I.1) that

$$P(6r+1,3) \le (6r)! - \sqrt{\frac{(6r)(6r-1)\cdots(5r+2)}{48r^3+32r^2+12r+1}}. \quad \text{(V.6)}$$

Since r is positive, we can assert that $m := \dfrac{(5r+2)(5r+3)(5r+4)}{48r^3+32r^2+12r+1} > 2.6$. Given that $n \ge 37$, it follows that $r \ge 6$. Therefore, we have

$$\sqrt{2.6}(5r+5)^{\frac{r-4}{2}} > \left\lceil\frac{6r+1}{3}\right\rceil + 2.$$

TABLE V
NEW $(7, d)$–CODES

d	$\lvert C_7^d \rvert$	Generators of H	$\lvert H \rvert$	λ_H	Elements of S_H
4	5565	[4, 2, 1, 3, 7, 5, 6] [3, 5, 2, 6, 7, 1, 4]	21	240	[1, 2, 3, 7, 4, 6, 5] [1, 2, 3, 5, 7, 6, 4] [1, 2, 6, 3, 4, 7, 5] [1, 2, 4, 7, 3, 5, 6] [1, 2, 5, 4, 3, 7, 6] [1, 2, 5, 6, 3, 4, 7] [1, 2, 4, 6, 5, 3, 7] [1, 2, 6, 7, 5, 3, 4] [1, 2, 7, 4, 6, 5, 3] [1, 4, 2, 3, 6, 7, 5] [1, 6, 2, 7, 4, 3, 5] [1, 5, 2, 7, 6, 3, 4] [1, 6, 4, 2, 3, 5, 7] [1, 6, 5, 2, 3, 7, 4]
5	3651	[3, 4, 1, 2, 6, 5, 7] [5, 2, 1, 7, 3, 4, 6]	42	57	[1, 2, 5, 3, 7, 6, 4] [1, 2, 7, 6, 4, 3, 5]
6	2811	[7, 2, 1, 5, 6, 4, 3] [3, 4, 2, 5, 7, 1, 6]	21	166	[1, 2, 3, 7, 6, 5, 4] [1, 2, 7, 4, 6, 5, 3] [1, 6, 2, 5, 4, 7, 3]
7	1684	[6, 2, 4, 3, 7, 1, 5] [1, 3, 6, 5, 7, 2, 4]	42	3	–
8	1181	[2, 5, 7, 3, 4, 1, 6]	7	624	[1, 2, 7, 6, 3, 5, 4] [1, 5, 7, 2, 4, 3, 6] [1, 5, 6, 3, 2, 7, 4]
9	686	[2, 5, 7, 4, 1, 3, 6]	3	1418	[1, 7, 2, 4, 6, 5, 3] [1, 6, 3, 4, 7, 5, 2] [4, 6, 2, 1, 7, 3, 5] [3, 6, 5, 1, 2, 7, 4]
10	475	[2, 4, 7, 5, 3, 6, 1]	6	92	[3, 6, 4, 2, 5, 1, 7]
11	219	[6, 5, 3, 7, 2, 1, 4]	2	1400	[1, 7, 3, 5, 6, 4, 2] [5, 4, 3, 2, 1, 7, 6] [7, 2, 1, 4, 6, 5, 3]
12	163	[2, 4, 7, 6, 3, 5, 1]	7	40	–
13	83	[1, 7, 6, 4, 5, 3, 2]	2	198	[6, 2, 4, 5, 7, 3, 1]
14	66	[6, 5, 3, 4, 2, 1, 7]	2	266	[7, 5, 1, 4, 3, 6, 2]

Hence, by substituting m with 2.6 in the inequality (V.6), we obtain

$$P(n,3) < (6r)! - \sqrt{2.6}(5r+5)^{\frac{r-4}{2}} < (6r)! - \left\lceil \frac{6r+1}{3} \right\rceil + 2.$$

Now, consider the case where $n \equiv 5 \pmod 6$. In this case, we have $r = \frac{n-5}{6}$. Again, from inequality (I.1), we find

$$P(n,3) \le (6r+4)! - 5\sqrt{\frac{(6r)(6r-1)\cdots(5r+6)}{48r^3+160r^2+250r+125}}. \quad \text{(V.7)}$$

Since r is positive, we have

$$s := \frac{(5r+6)(5r+7)(5r+8)}{48r^3+160r^2+250r+125} > 2.6.$$

Thus, considering $r \ge 6$ and replacing s with 2.6 in the inequality (V.7), we can conclude:

$$\begin{aligned} P(n,3) &< (6r+4)! - 5\sqrt{2.6}(5r+9)^{\frac{r-4}{2}} \\ &< (6r+4)! - \left\lceil \frac{6r+5}{3} \right\rceil + 2. \end{aligned}$$

In view of [1, Theorem 1.1], the proof is complete. □

TABLE VI
NEW $(8, d)$–CODES

d	$\lvert C_8^d \rvert$	Generators of H	$\lvert H \rvert$	λ_H	Elements of S_H
3	105236	[4, 2, 5, 1, 3, 8, 7, 6] [8, 6, 3, 4, 1, 7, 2, 5]	336	120	[1, 2, 3, 4, 5, 8, 7, 6] [1, 2, 3, 4, 6, 8, 5, 7] [1, 2, 3, 7, 4, 5, 6, 8] [1, 2, 3, 8, 4, 7, 5, 6] [1, 2, 3, 7, 4, 8, 6, 5] [1, 2, 3, 6, 5, 4, 8, 7] [1, 2, 3, 8, 5, 4, 6, 7] [1, 2, 3, 8, 6, 4, 5, 7] [1, 2, 3, 7, 5, 8, 4, 6] [1, 2, 3, 7, 8, 6, 5, 4]
4	89682	[7, 1, 8, 3, 4, 2, 6, 5] [6, 5, 4, 2, 3, 8, 1, 7]	168	240	[1, 2, 3, 4, 8, 5, 7, 6] [1, 2, 3, 4, 6, 8, 7, 5] [1, 2, 3, 7, 4, 5, 8, 6] [1, 2, 3, 5, 8, 4, 6, 7] [1, 2, 3, 6, 5, 4, 8, 7] [1, 2, 3, 6, 7, 4, 5, 8] [1, 2, 3, 8, 7, 4, 6, 5] [1, 2, 3, 5, 7, 6, 4, 8] [1, 2, 3, 8, 5, 7, 6, 4] [1, 2, 3, 6, 8, 7, 5, 4] [1, 2, 5, 4, 3, 6, 8, 7] [1, 2, 8, 5, 4, 3, 6, 7]
5	66442	[7, 2, 8, 6, 5, 4, 1, 3] [6, 4, 3, 5, 2, 8, 7, 1]	336	16	[1, 2, 3, 8, 4, 7, 5, 6]
6	54709	[8, 3, 4, 6, 5, 7, 1, 2] [5, 2, 4, 8, 3, 1, 6, 7]	56	672	[1, 2, 3, 8, 4, 6, 7, 5] [1, 2, 3, 7, 6, 4, 8, 5] [1, 2, 3, 5, 8, 6, 7, 4] [1, 2, 6, 5, 3, 4, 8, 7] [1, 2, 7, 5, 3, 6, 8, 4] [1, 2, 7, 8, 6, 3, 5, 4]
7	37499	[8, 5, 4, 1, 6, 3, 7, 2] [7, 2, 1, 3, 6, 8, 5, 4]	56	390	[1, 2, 7, 6, 3, 4, 5, 8] [1, 2, 4, 6, 7, 8, 5, 3]
8	29249	[7, 2, 1, 8, 5, 4, 3, 6] [1, 3, 8, 7, 5, 4, 6, 2]	21	900	[5, 2, 3, 4, 1, 7, 6, 8] [6, 5, 2, 3, 1, 4, 8, 7] [2, 6, 7, 5, 1, 4, 8, 3] [6, 2, 7, 3, 1, 4, 8, 5] [7, 2, 6, 8, 1, 3, 5, 4]
9	18352	[3, 7, 1, 6, 8, 4, 2, 5] [3, 5, 4, 8, 6, 1, 2, 7]	16	462	[1, 2, 6, 8, 4, 3, 7, 5] [1, 6, 5, 2, 7, 4, 3, 8] [1, 7, 4, 3, 2, 6, 5, 8]
10	13529	[7, 2, 5, 6, 8, 1, 3, 4]	7	108	[1, 2, 7, 3, 8, 6, 5, 4] [1, 2, 6, 8, 4, 5, 7, 3] [7, 1, 3, 4, 5, 8, 6, 2] [6, 1, 3, 5, 2, 8, 4, 7] [8, 1, 3, 5, 7, 2, 6, 4] [5, 1, 4, 7, 3, 2, 6, 8]
11	8135	[2, 7, 5, 8, 4, 6, 3, 1]	7	2520	[8, 6, 7, 2, 3, 1, 4, 5] [4, 3, 7, 2, 5, 1, 6, 8] [8, 4, 6, 5, 7, 1, 2, 3]
12	6163	[7, 8, 5, 6, 3, 4, 1, 2] [4, 8, 5, 2, 7, 3, 6, 1]	24	12	–
13	3169	[4, 6, 1, 5, 8, 3, 7, 2]	7	708	[3, 7, 4, 6, 2, 5, 1, 8]
14	2324	[4, 6, 1, 5, 8, 3, 7, 2]	7	708	[3, 7, 4, 6, 2, 5, 1, 8]
15	810	[4, 6, 7, 8, 1, 3, 5, 2]	8	168	–
16	607	[4, 7, 6, 8, 1, 2, 3, 5] [7, 4, 5, 3, 2, 1, 8, 6]	8	96	–
17	252	[7, 3, 8, 2, 6, 1, 5, 4]	4	112	–
18	189	[7, 3, 8, 2, 6, 1, 5, 4]	4	112	–

VI. CONCLUSION

In this study, we focused on permutation codes under the Kendall τ-metric due to their relevance in applications such as flash memories and DNA-based data storage. We established new theoretical results, including an exact value of $P(n, d)$ for $\frac{3}{5}\binom{n}{2} < d \le \frac{2}{3}\binom{n}{2}$, and introduced improved constructions for small values of n by forming permutation codes from subgroups of S_n and their cosets. Moreover, we derived a novel upper bound on $P(n, 3)$ that surpasses existing bounds for all prime numbers $n \ge 37$.

APPENDIX

The information presented in Tables III, IV, V, and VI includes the generators of the subgroup H in S_n (i.e., a subset of elements of H such that every element of H can be expressed as a combination of finitely many elements from this subset and their inverses) and the set S_H for $n \in \{5, 6, 7, 8\}$, as derived from Algorithm 1, alongside additional software verification details. In these tables, C_n^d denotes the size of the set of all subgroups of S_n that are (n, d)-PCs under the Kendall τ-metric, and λ_H represents the number of left cosets of H that are (n, d)-codes under the Kendall τ-metric. In fact, in these tables, for each pair (n, d), the union $\mathcal{C} := \cup_{x \in S_H \cup \{\xi\}} xH$ forms a new (n, d)-PC. Tables III and IV present the size of $|\mathcal{C}|$ obtained using Algorithm 1, alongside the value of $P(n, d)$. These results demonstrate that the size of the (n, d)-permutation codes generated by Algorithm 1, for $n \in \{5, 6\}$, is either equal to or very close to $P(n, d)$.

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